On Generation of Error Vectors[1]

Otokar Grošek, Viliam Hromada

Institute of Computer Science and Mathematics
Slovak University of Technology

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Motivation

- Generation of n-bit vectors with Hamming weight t (constant weight codewords, error vectors) is an important part of various cryptosystems.
  - E.g. in McEliece cryptosystem, we need to generate a random error vector of a given length n and with a specific Hamming weight t.
  - In Niederreiter cryptosystem, not only we need to have a constant weight codeword, we also want to be able to convert it to a binary string and vice versa.
- There are many proposed algorithms.
- We propose a new method, interesting from an algebraic point of view.

Generation of a random n-bit vector with t ones

There are many ways, how to uniformly generate a random error vector with a good pseudorandom generator, e.g.:

- **Knuth shuffling** - start with a n-bit vector $\underbrace{11...1}_{t}\underbrace{0...0}_{n-t}$ and t-times swap one of the ones with a bit at a random position in the vector.
- $n = 5, t = 2 : w = 11000$
  1. $r = 3, r \in_R \{1, ..., 5\} : 01100$
  2. $r = 5, r \in_R \{2, ..., 5\} : 00101$
- Let $w[i]$ be the i-th bit of the vector w.
- It can be proved that the probability $Pr(w[i] = 1) = \frac{t}{n}$ for any $i \in \{1, ..., n\}$ in the resulting vector.
- The assumption is that we are able to uniformly select numbers at random modulo different integers.

Generation of a random n-bit vector with t ones

- **Permutation of n elements** - a permutation of n elements can be used to determine the positions of one bits.
- $n = 5, t = 3 : w = 00000, C = \{1, 2, 3, 4, 5\}$
  1. $r = 3, r \in_R \{1, ..., 5\} : C[r] = 3, 00100, C = \{1, 2, 5, 4, 5\}$
  2. $r = 3, r \in_R \{1, ..., 4\} : C[r] = 5, 00101, C = \{1, 2, 4, 4, 5\}$
  3. $r = 2, r \in_R \{1, ..., 3\} : C[r] = 2, 01101, C = \{1, 4, 4, 4, 5\}$
- The assumption is that we are able to uniformly select numbers at random modulo different integers.

Generation of a random n-bit vector with t ones

- **Trellis generator** (proposed as suitable for hardware implementation by Butler, Sasao in 2011)
- The resulting vector is generated in n clocks - bit by bit.
- The probabilities of generating a bit 0/1 constantly changes according to the previously generated bits, so that all possible vectors have the same probability.
- In the beginning, the probability of a bit 1 is $\frac{t}{n}$.
  - Bit 0 => the probability changes to $\frac{t}{n-1}$.
  - Bit 1 => the probability changes to $\frac{t-1}{n-1}$.
- Easy way is to generate a number from a certain set and have a threshold for bits 0/1.
- The assumption is that we are able to uniformly select numbers at random modulo different integers.

Generation of a random n-bit vector with t ones - Trellis

$n = 5, t = 2$. Green are numbers associated with 0 bit, red are numbers associated with 1 bit.

1. $\{1, 2, 3, 4, 5\}$. $r \in_R \{1, ..., 5\}, r = 2 \Rightarrow w = 0$
2. $\{1, 2, 3, 4\}$. $r \in_R \{1, ..., 4\}, r = 1 \Rightarrow w = 00$
3. $\{1, 2, 3\}$. $r \in_R \{1, ..., 3\}, r = 3 \Rightarrow w = 001$
4. $\{1, 2\}$. $r \in_R \{1, ..., 2\}, r = 2 \Rightarrow w = 0011$
5. $\{1\}$. $r \in_R \{1, ..., 1\}, r = 1 \Rightarrow w = 00110$

Index to constant weight codeword converter

- More advanced methods are able to generate a constant weight codeword from a binary sequence / integer.
- And also convert the constant weight codeword back to the original piece of information.
- One of the well-known methods is an algorithm based on Combinatorial Number System (Cover, 1973).
- A different one, based on run-length encoding, was proposed by Sendrier in 2005.

CNS

- There are $\binom{n}{t}$ n-bit words with Hamming weight t.
- Every number N from 0 to $\binom{n}{t}$ can be expressed by a Combinatorial Number System, i.e.

$$N = \binom{c_t}{t} + \binom{c_{t-1}}{t-1} + \ldots + \binom{c_1}{1},$$

  where $c_t > c_{t-1} > \ldots > c1 \geq 0$.
- The set of digits c_i is unique for each N.
- Digits c_i represent positions of 1 bits in a n-bit vector, which defines a bijective mapping between constant weight codewords of length n, weight t and integers ranging from 0 to $\binom{n}{t} - 1$.

CNS

$n = 5, t = 2$. There are $\binom{5}{2} = 10$ different vectors.

N	Representation	Codeword
0	$\binom{1}{2} + \binom{0}{1}$	00011
1	$\binom{2}{2} + \binom{0}{1}$	00101
2	$\binom{2}{2} + \binom{1}{1}$	00110
3	$\binom{3}{2} + \binom{0}{1}$	01001
4	$\binom{3}{2} + \binom{1}{1}$	01010
5	$\binom{3}{2} + \binom{2}{1}$	01100
6	$\binom{4}{2} + \binom{0}{1}$	10001
7	$\binom{4}{2} + \binom{1}{1}$	10010
8	$\binom{4}{2} + \binom{2}{1}$	10100
9	$\binom{4}{2} + \binom{3}{1}$	11000

Run-length encoding

- Sendrier proposed a way how to encode constant weight codewords into binary strings (and vice-versa).
- It is a modified Golomb's run-length encoding.
- The original encoding scheme uses a parameter d which depends on the probability that a memoryless source generates a bit 0 (or 1).
- Then, rather than encoding some binary sequence, it encodes the lengths of runs of zero bits.
- Sendrier's modification changes the parameter d after the encoding of each run and is used to encode the constant weight codewords into binary strings.
- One drawback of this method is a variable length of the resulting binary strings.

Run-length encoding

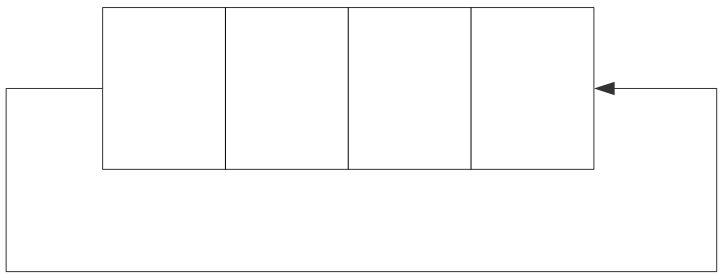
$n = 5, t = 2$.

Codeword	Run-length	Binary sequence
00011	[3 0]	11
00101	[2 1]	1001
00110	[2 0]	10000
01001	[1 2]	011
01010	[1 1]	0101
01100	[1 0]	0100
10001	[0 3]	0011
10010	[0 2]	00100
10100	[0 1]	0001
11000	[0 0]	0000

New proposed method

- We propose a new method, which is not very practical, but is interesting from an algebraic point-of-view.
- Let $V_n = \mathbb{F}_2^n$ be the n-dimensional vector space and $E_t = \{e | hwt(e) = t\} \subset V_n$ be the set of all binary vectors of length n and weight t.
- Let **A** be the associated $n \times n$ matrix to the characteristic polynomial $f(x) = x^n + 1$ over $\mathbb{F}_2$ of the LFSR.

- $f(x) = x^n + 1$ is a characteristic polynomial of a LFSR whose new feedback bit is equal to its output bit.
- Since its length is n, the period of its output is also at most n.
- E.g. a polynomial $f(x) = x^4 + 1$ corresponds to a LFSR:



- Its associated matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $[u] = \{u, u\mathbf{A}, u\mathbf{A}^2, ..., u\mathbf{A}^{d-1}\}$ be a class of words obtained from u by consecutive shifts, where d is the smallest period.

$$\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}\mathbf{A} = \begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\mathbf{A} = \begin{pmatrix}0 & 0 & 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 1 & 1\end{pmatrix}\mathbf{A} = \begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix}\mathbf{A} = \begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}$$

$$[1100] = \{1100, 1001, 0011, 0110\}$$

$$[1010] = \{1010, 0101\}$$

- We define a relation ρ_n such that $u\rho_n v$ iff u, v belong to the same class.
- This relation is an equivalence relation.
- $u\rho_n = [u]$.
- For any u the cardinality of $[u] = u\rho_n$ divides the order of $\mathbf{A}$ in the general linear group $GL(n, \mathbb{F}_2)$.

Theorem

Let ρ_n be the equivalence relation on E_t defined such that $u \rho_n v$ iff both u, v belong to the same class. Then

1. *There exists a class $u\rho_n$ with cardinality r if and only if $r \mid n$ and $\frac{n}{r} \mid t$.*
2. *All classes have the same cardinality if and only if $gcd(n, t) = 1$.*

Example

Let $n = 20, t = 10$. Then we have the following classes:

- 9225 classes with cardinality $r = 20$,
- 25 classes with cardinality $r = 10$,
- 1 class with cardinality $r = 4$,
- 1 class with cardinality $r = 2$.

Classes with a given cardinality

- Let $C(n, t, r)$ denote the number of classes with cardinality r for given n and t.
- E.g. in the previous example $C(20, 10, 2) = 1, C(20, 10, 20) = 9225$.
- According to the theorem and definition of ρ_n:

$$\binom{n}{t} = \sum_{\substack{r|n \\ n/r|t}} rC(n, t, r). \tag{1}$$

- 2 special cases:

$$C(n, n, r) = \begin{cases} 1, & \text{if } r = 1; \\ 0, & \text{if } r > 1; \end{cases} \qquad C(n, 0, r) = \begin{cases} 1, & \text{if } r = 1; \\ 0, & \text{if } r > 1. \end{cases}$$

Classes with a given cardinality

Theorem

Let for given n, t, $0 < t < n$, be an integer r, $1 < r < n$, such that $r \mid n$ and $\frac{n}{r} \mid t$. Then there exists at least one class $|u\rho_n| = r$ and word w of the length r and weight $\frac{tr}{n}$ such that

$$u = \overbrace{w||w||...||w}^{n/r} = w^{n/r}$$

Moreover, let $d = gcd(n, t)$ and $d(g)$ be the number of divisors of g. Then we have precisely $d(g)$ nonzero summands in the formula (1).

Examples

- $n = 6, t = 6$. This is one of the special cases, since $n = t$.
  $\binom{6}{6} = 1C(6, 6, 1) = 1 \times 1$.
- $n = 20, t = 10 \Rightarrow gcd(20, 10) = 10, d(10) = 4$
  $\binom{20}{10} = 20C(20, 10, 20) + 10C(20, 10, 10) + 4C(20, 10, 4) + 2C(20, 10, 2) = 20 \times 9225 + 10 \times 25 + 4 \times 1 + 2 \times 1$
- $n = 1024, t = 70 \Rightarrow gcd(1024, 70) = 2, d(2) = 2$.
  $\binom{1024}{70} = 1024C(1024, 70, 1024) + 512C(1024, 70, 512)$.
  In the case $r = 512, u = w||w$.
  $512C(1024, 70, 512) = \binom{512}{35} \Rightarrow C(1024, 70, 512) = \frac{\binom{512}{35}}{512}$
  $C(1024, 70, 1024) = \frac{\binom{1024}{70} - \binom{512}{35}}{1024}$

Generation of a random constant weight word

- We can use the fact that all constant weight words of length n and weight t are divided into equivalence classes with r elements.
- First, randomly select one of the equivalence classes.
- Second, randomly select one of the elements from the equivalence class.
- It may happen that different classes have different number of elements.
- So that the probability distribution of selecting a particular class has to be proportionate to the cardinalities.

Example

Considering $n = 20, t = 10$, one may choose:

- an equivalence class with cardinality $r = 20$ with probability $\frac{20}{\binom{20}{10}}$.
- an equivalence class with cardinality $r = 10$ with probability $\frac{10}{\binom{20}{10}}$.
- an equivalence class with cardinality $r = 4$ with probability $\frac{4}{\binom{20}{10}}$.
- an equivalence class with cardinality $r = 2$ with probability $\frac{2}{\binom{20}{10}}$.

And afterwards a random number from the set $\{0, ..., r - 1\}$ to get the corresponding shift of the representative of the class.

Example

- For a class $[00110101010101010101]\rho_n$ with $r = 20$, if a random shift 3 is generated then the resulting vector is 10101010101010101001.
- For a class $[00110011001100110011]\rho_n$ with $r = 4$, if a random shift 1 is generated then the resulting vector is 01100110011001100110.

Thank you for your attention!